Assessing and Improving Helioseismological Inferences of Solar Structure

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Abstract:

Acoustic vibrations of the Sun contain information about the distributions of soundspeed, material density, and angular velocity within the Sun. Current methods for imaging these properties of the solar interior from helioseismic eigenfrequencies produce images that might resemble the Sun, but the formal uncertainties of the techniques do not accurately quantify possible differences between the models and the truth. Nor can the methods incorporate physical constraints that might reduce the uncertainty. The effects of data errors, modeling algorithms, and theoretical approximations on the reliability of the images are largely unexplored; published uncertainties approximate the variability of the estimation method conditional on a particular parametric representation of the solar property, and a host of other assumptions, without regard for possible biases. The proposed research is to develop and apply methods to study both the variability and the bias of estimates of the distribution of soundspeed in the solar interior and the distribution of angular velocity in the solar interior, under less restrictive assumptions, incorporating prior physical constraints such as bounds on energy or rotationally-induced oblateness to control the bias.

Shorter, but still conservative, confidence intervals for solar properties can be found by using unusual functionals for measuring fit to the data, rather than conventional l_p misfit norms such as the common sum of squared residuals. Such alternatives will be explored and applied to find simultaneous confidence envelopes for the distribution of soundspeed in the solar interior, incorporating uncertainties due to surface effects and to neglected higher-order terms in the asymptotics, and to make direct inferences about spatial variations in the distribution of angular velocity within the Sun from the splitting of normal mode frequencies. A fringe benefit of the new misfit measures will be "teleological" data compression techniques that enhance

the ability to image the solar interior from normal mode data.

Proposed Research.

1) Testing Hypotheses About Rotation.

Eigenmodes of the Sun with the same radial order and angular degree would have the same eigenfrequency if the Sun were spherically symmetric. Rotation of the Sun is one origin of asymmetry that "splits" degenerate eigenfrequencies. To first order, the odd functional part of the splitting of individual singlets from the central multiplet frequency is a linear function of the angular velocity; asymmetry in the acoustic velocity structure contributes to the even part of the splitting. Observations of the splitting of normal mode frequencies can and have been used to estimate the distribution of angular velocity within the Sun. It is common practice in these studies to take the soundspeed distribution within the Sun as known; within that approximation, the splitting is linearly related to the angular-velocity distribution, to first order in the angular velocity. Let δ_j denote the jth observed frequency splitting, ϵ_j the observational error in its measurement, Ω the distribution of angular velocity within the Sun, and B the Sun. Within the customary approximation,

$$\delta_j = \int_B K_j(\mathbf{r})\Omega(\mathbf{r})d\mathbf{r} + \epsilon_j, \quad j = 1, \ldots, n. \tag{1}$$

It is often assumed (and reported by observers) that the errors ϵ_j are independent, zero mean random variables with known variances σ_j^2.

2) Tailored Kernels by Numerical Optimization

Fitted models of solar rotation have many intriguing features. However, since the error analysis is not complete, it is not possible to infer from a feature of a fitted model whether or not the Sun is likely to possess the same feature. Genovese, Stark and Thompson (Ap.J., 1995) show that from Big Bear Solar Observatory (BBSO) data (Woodard and Libbrecht (Ap.J., 1993), even with very restrictive assumptions, inferences based on pointwise estimates of the rotation rate can not even rule out solid-body rotation. To make sound inferences requires abandoning the search for entire rotation models.

Gough, Sekii and Stark (Ap.J., 1995, submitted) present a variety of ways to make simultaneous inferences about a collection of linear functionals of solar structure, and illustrate a promising approach I propose to pursue: to make inferences about the solar interior by constructing functionals that are directly sensitive to hypotheses of interest. One may then test the hypotheses by finding confidence intervals for the functionals.

For example, suppose we wish to determine whether or not the average rotation rate increases with depth across the convection zone. For a given set of n real numbers $\{\beta_j\}_{j=1}^n$, define the linear combination

$$f_\beta(\mathbf{r}) \equiv \sum_{j=1}^n \beta_j K_j(\mathbf{r}).$$

Let B^- and B^+ be non-overlapping spherical shells such that the outer radius of B^- is less than the inner radius of B^+. Suppose we could find a set of coefficients such that $f_\beta(\mathbf{r})$ satisfied

$$f_\beta(\mathbf{r}) \leq 0, \quad \mathbf{r} \in B^-, \tag{2}$$

$$f_\beta(\mathbf{r}) \geq 0, \quad \mathbf{r} \in B^+, \tag{3}$$

$$f_\beta(\mathbf{r}) = 0, \quad \mathbf{r} \in B^0, \tag{4}$$

$$\int_{B^-} f_\beta(\mathbf{r}) d\mathbf{r} = -1, \tag{5}$$

and

$$\int_{B^+} f_\beta(\mathbf{r}) d\mathbf{r} = 1. \tag{6}$$

Then $\int_B f(\mathbf{r})\Omega(\mathbf{r})d\mathbf{r} < 0$ implies that in some subset of B^-, Ω is larger than it is anywhere on B^+, and hence that the rotation rate must somewhere increase with depth. The linear combination of the data $\sum_{j=1}^n \delta_j$ is an unbiased estimate of $\int_B f(\mathbf{r})\Omega(\mathbf{r})d\mathbf{r}$ with variance $\sum_j \beta_j^2\sigma_j^2$, and we can readily construct a confidence interval for $\int_B f(\mathbf{r})\Omega(\mathbf{r})d\mathbf{r}$. It remains to find the coefficients $\{\beta_j\}$. If more than one set of coefficients satisfies the restrictions (**??** - **??**), we would like the one that minimizes $\sum_j \beta_j^2\sigma_j^2$, as the resulting confidence interval would be shortest.

We are thus led to the optimization problem:

$$\min \sum_j \beta_j^2\sigma_j^2 \tag{7}$$

subject to (**??** - **??**), which is a semi-infinite quadratic program. This problem can be solved using the algorithm given by Stark and Parker (Comp. Stat., 1995) using a strategy of successive refinement of a mesh of constraints.

The linear inequality constraints can be imposed at a grid of points, and the corresponding finite-dimensional quadratic programming problem solved. One may search the solution to see whether it violates the constraints at points not in the grid. If so, new points can be added to the grid near the violation, and the resulting finite-dimensional problem solved. This can be repeated until the constraints are satisfied off the grid as well as on. Solutions at the successive levels should be similar, so the cost of additional iterations is expected to be modest. The dimension of the problem can be further reduced by using only a subset of the data mappings, and adding new ones using a greedy criterion that checks how nearly collinear the violation of the constraint is with the function to be added.

Sekii, Genovese, Gough and Stark (ESA, 1995) and Gough, Sekii, and Stark (Ap.J., 1995, submitted) show using averages of the Big Bear Solar Observatory (BBSO) data for the years 1986 and 1988-1990 that it is possible to find linear combinations with isolated sign changes in colatitude, using the SOLA method described by Pijpers and Thompson (Astron. Astroph., 1992, 1994) They used a fairly crude construction of the kernels that I propose to improve using the optimization formulation presented above. Applying this approach to larger data sets from SOHO-SOI should yield precise tests of hypotheses about the Sun's differential rotation.

The optimization problem is very memory-intensive, as the sign constraints need to be imposed at a relatively dense grid in the solar interior. The problem that must be solved is a sequence of quadratic programs, with a large number of variables (up to 10,000) and many more constraints. Many of the constraints will be redundant, but *a priori* it is hard to tell which. Finding an efficient means of solving this problem will be an important part of the work. At present, one promising approach is to send subsets of the data mappings to many different processors on the Statistical Computing Facility (SCF) network, and to combine and winnow subsets using a measure of merit that includes the degree to which the constraints are violated and the variance of the resulting functional f_β. As each processor completes its subproblem, it would be allocated a new subset of the mappings constructed from the most successful so far. This strategy should lend itself well to distributed computing, as each processor has a lot of work to do between times it communicates with others; local experience suggests that inter-process communication is the most severe bottleneck in obtaining linear scaling of speed with the number of machines employed. After a screening process like this one reduces the number of potentially useful data mappings to a reasonable number, those that remain can be used in a larger optimization problem on a single machine with a large amount of memory. In this way I hope to find a nearly optimal solution.

3) Physical Constraints in Rotation Estimates.

There are a number of physical constraints that can be used to improve the reliability of rotation estimates. For example, the rotational kinetic energy is a quadratic functional of the rotation, and clearly must be less than the rest mass of the Sun. While such a constraint might seem hopelessly loose, Backus (Geophys. J., 1989) and Stark (Geophys. J. Int., 1992) show that a similarly prima facie useless constraint, that the energy stored in Earth's magnetic field is less than Earth's rest mass, is useful in estimating spherical harmonic components of Earth's magnetic field. Other physical constraints one might use include the observed oblateness (another quadratic functional), the constraint that rotation does not cause particle velocities to exceed the speed of light, or the constraint that matter is gravitationally bound. The last two constraints are pointwise linear inequalities. Stark (J. Geophys. Res., 1992) shows how to use both quadratic and linear inequalities to reduce the uncertainties in inverse problems. With Christopher Genovese (Carnegie-Mellon University) and Douglas Gough (University of Cambridge) I have begun to explore these constraints to test hypotheses about the distribution of angular velocity within the Sun by seeking models that violate various physical hypotheses while satisfying the constraint and fitting the data. We have developed theory that allows us to take into account approximation errors from representing the solar rotation as a finite linear combination of known functions; the theory appears to extend to a wide variety of hypothesis-testing problems in infinite-dimensional spaces. I propose to continue this work and perform the computations using the oblateness constraints to make inferences from SOHO-SOI data.

Applying these constraints will requires special efficient algorithms for nonsmooth convex optimization problems in many variables, algorithms that exploit the special structure of the problems. A promising approach is to use a multistage algorithm that starts with subsets of the data, keeping the dimension of the problem low, then successively refines the solution until all data are included. Tests of different hypotheses could be distributed efficiently over the SCF network, as the resulting problems share the same data but do not need inter-process communication.

4) Asymptotic Inversion for Radial Soundspeed Structure

Let ω_{kl} be the eigenfrequency with radial number k and angular number l. It can be shown (see Brodsky and Levshin, Geophys. J. Roy. astr. Soc., 1979) that

$$\frac{\pi(k+\alpha)}{\omega_{kl}} = \int_{r_t}^{R} \left(\frac{r^2}{c^2(r)} - \left(\frac{l+1/2}{\omega_{kl}} \right)^2 \right)^{1/2} r^{-1} dr + O(\omega_{kl}^{-2}), \tag{8}$$

where R is the solar radius, r is distance from the center of Sun, $c(r)$ is the soundspeed at radius r, and α is the phase shift factor that captures the effect of the boundary condition at $r = R$. Equation (**??**) is sometimes called the Duvall equation (after Tom Duvall). To first order in ω_{kl}^{-1}, this relation is equivalent to Abel's equation. The asymptotic inversion of free-oscillation data via Abel's equation was first done for Earth by Brodsky and Levshin (the first paper in English is Geophys. J. Roy. astr. Soc., 1979). The technique has also been used for the inversion of solar free-oscillation data by Christensen-Dalsgaard and others (Nature, 1985), and by Brodsky and Vorontsov (IAU, 1988).

These studies invoke an analytic inversion formula that, if the data were perfect and complete (if a continuum of normal modes were observed, without observational errors), would find the unique one-dimensional soundspeed profile consistent with the observations, within the asymptotic theory. Unfortunately, there is only a finite (though quite large) number of data, and the data contain observational errors. In this circumstance an infinite number of soundspeed profiles are consistent with the observations; which of these is found by a particular study depends critically on the interpolation scheme used to construct a continuous curve from the noisy observations—different interpolations can produce radically different models of soundspeed. The previous studies construct individual models consistent with the observations, but as there are infinitely many such models, properties of any particular model may not be shared by the actual solar soundspeed profile. There is no reasonable error analysis of this inversion procedure.

Garmany (Geophys. Res. Lett., 1979) proposed a method for finding an envelope containing all solutions to Abel's problem that fit the data adequately. His work was extended to spherical geometry and to incorporate additional constraints on monotonicity of seismic velocity with depth by Stark *et al.* (J. Geophys. Res., 1986), and to different measures of misfit by Stark and Parker (J. Geophys. Res., 1987). The theoretical underpinnings of this general approach to inference are set out by Stark (J. Geophys. Res., 1992), who also addresses the problem of solving the infinite-dimensional problem rigorously, and some extensions relevant to helioseismology. See also Lang (IEEE-IT, 1985) and Stark (Geophys. J. Rpoy. astr. Soc., 1987).

This "strict bounds" approach uses the data, the statistics of the data errors, the mathematical relation between the data and the unknown soundspeed profile, and physical constraints such as monotonicity of the soundspeed profile, to define a confidence region for the true soundspeed profile within the infinite-dimensional space of all soundspeed profiles. Properties of that set are, with a specified probability, properties of the true soundspeed. By solving optimization problems to find the smallest and largest values "interesting" functionals attain within the confidence region, we get confidence intervals for functionals of the true soundspeed. In particular, one can find confidence intervals for the depth at which a certain soundspeed occurs. This can be done, with simultaneous coverage probability, at an arbitrary number of soundspeeds. The result is that one may construct a "confidence envelope" that, with specified probability, contains the true soundspeed profile. It is possible to do this quite rigorously, incorporating the true infinite-dimensional uncertainty, the data errors, modeling errors such as neglected terms in the asymptotics, *etc.* For example, in equation (**??**), both the limit of integration and integrand depend on ω_{kl}, which is not known precisely—only estimates with errors are available. Thus we do not even know precisely what functional of the Sun was measured. I call this the "fuzzy functional" problem; Stark (J. Geophys. Res., 1992) shows how to incorporate this additional uncertainty to get a conservative confidence envelope for soundspeed. Expressions for the higher-order terms in the asymptotic relations, including perturbations to gravity, have been derived recently by Brodsky and Vorontsov (Ap. J., 1993); by evaluating these expressions for plausible solar structures we can obtain reasonable bounds on the neglected higher-order terms. The "strict bounds" approach can incorporate this additional source of uncertainty as well.

The usual misfit criterion used to define a confidence region for the noise-free observations is based on an l_p norm of the errors, with p equal to 1, 2, or ∞. (The chi-square, or 2-norm measure is the most common.) This procedure is not statistically consistent: as the number of data grows, the confidence intervals for a given functional do not shrink to zero (Stark, Phys.

Earth. Planet. Int., 1995, in prep.) D.L. Donoho (personal communication, 1994) has proposed using a wavelet-based measure of misfit instead of an l_p norm (Stark, Herron, and Matteson, Appl. Spect., 1993, use such a measure), and claims that the wavelet misfit measure produces confidence intervals whose widths shrink to zero at essentially the asymptotically optimal rate as the number of data grows, and that automatically adapts to the inherent smoothness of the object. Work by Stark also suggests that simply averaging sets of observations that sample the Sun in similar ways can also lead to optimal rates of convergence of the confidence intervals. Benjamini and Stark (J. Amer. Stat. Assoc., 1996, in press) show that unusual geometries for confidence sets can be employed to improve tests of multiple hypotheses, for example, improving the ability to distinguish several effects from zero simultaneously.

I propose to explore and apply these approaches to estimate radially-averaged soundspeed within the Sun, incorporating the effects of data errors, asymptotic approximations, and "fuzzy functionals" into the analysis of the uncertainty of the results. The new misfit measures give rise to semi-infinite linear or quadratic programs. Stark (J. Geophys. Res., 1992) discusses some strategies for solving such problems computationally, and Stark and Parker (Comp. Stat., 1995) present an algorithm for solving the subproblems that arise. The dual approach is particularly well-suited for distributed computations on the SCF network.

An important fringe benefit of this approach, which should extend to other inverse problems as well, is that the new misfit functionals afford a substantial "compression" of the observations, while enhancing our ability to recover solar structure. For example, Stark (Phys. Earth Planet. Inter., 1995, in prep.) proves that in a particular nonparametric regression problem, it is preferable to compute with $n^{1/3}$ averages of the data than with the n original data: the derived confidence intervals are shorter, and the computational savings are enormous. Hengartner and Stark (Ann. Stat., 1995) proved that a similar phenomenon holds when estimating probability density functions (using a collection of weighted averages that number only a fractional power of the observations gives optimally narrow confidence intervals); and Stark, Herron and Matteson (Appl. Spect., 1993) found that a reduction of 3601 data to 90 weighted averages (specially chosen wavelet coefficients) improved the recovery of rock mineralogy from Fourier-Transform infra-red spectroscopic data.

Exploring such "teleological" data compression, where the goal is to recover solar structure, rather than to reconstruct the original observations, is perhaps the most important part of the planned research in the long-term. As the number of high-quality observations of solar free-oscillations grows through SOHO-SOI, the computational savings will become crucial if the helioseismic community is to extract as much information as possible about the solar interior.

5) Estimating the Surface Phase Shift α.

In order to apply the analytic inversion or strict bounds, it is necessary to estimate the "surface phase shift" α, which is a function of frequency ω. The phase shift is related to boundary conditions at the solar "surface"—see equation **??**. The inability to remove completely the effects of α from the normal mode data limits the residual noise level in the data used in asymptotic inversions; the "strict bounds" formulation can incorporate this source of uncertainty. The ratio $\frac{\pi(k+\alpha)}{\omega_{kl}}$ is known to be monotonic *a priori.* This monotonicity can be exploited to reduce the uncertainty in estimates of α. Monotonicity constraints are a special case of pointwise linear inequality constraints, as they constrain the generalized derivative to be of one sign everywhere; thus, studying this problem should help us in more complicated situations such as using the constraint that matter is gravitationally bound in estimating the helioseismic rotation.

The current method for determining α is to fit a simple function to the data in such a way as to minimize the scatter of the reduced data. The method does not provide uncertainty estimates. The data are insensitive to shifts of the entire α curve, so without additional constraints, only changes in α as a function of frequency are identifiable.

Many authors have studied the monotone regression problem (see Robertson *et al.* (John Wiley and Sons, 1988) for an overview; see Mukarjee and Stern (J. Amer. Stat. Assoc., 1994) for a recent development and references), which can be summarized as follows: We observe n data δ_j, $j = 1, \ldots, n$, that are samples of a monotone function f, corrupted by additive Gaussian noise:

$$\delta_j = f(t_j) + \epsilon_j, \quad j = 1, \ldots, n, \tag{9}$$

where $\{\epsilon_j\}$ are independent, identically distributed, zero-mean Gaussian random variables with unit variance. Various techniques have been proposed for estimating f from these observations, but none gives conservative confidence intervals for f nor a simultaneous confidence envelope for f.

Hengartner and Stark (Ann. Stat., 1995) develop a technique for estimating probability density functions nonparametrically, subject to a shape restriction on the density, for example, the constraint that the density is monotone or has at most k modes. Their method gives a conservative confidence envelope for the density whose width converges to zero at the optimal rate at all points where the density is continuous. The basic idea behind their method is to set up a nonparametric confidence region for the unknown density based on a distribution-free test, then solve optimization problems subject to the shape restriction on the density to find the largest and smallest values any density can have and adequately fit the data and satisfy the shape restriction. This is, in essence, the same as the "strict bounds" approach outlined above. The optimization problems can be reduced exactly to finite-dimensional linear programs.

Their general approach can be adapted to the monotone regression problem by considering a confidence set for the entirety of f based on a measure of misfit to the data $\delta = (\delta_1, \ldots, \delta_n)$, intersecting that set with the set of monotone functions, and solving a constrained optimization problem to find the largest and smallest values at the point t_0 among functions in that intersection. The result is a conservative confidence interval for $f(t_0)$, and as t_0 varies, the confidence intervals have simultaneous coverage probability. The intervals can be interpolated conservatively using the monotonicity constraint to obtain a confidence envelope for the entire function f; *i.e.*, a pair of functions $\ell(t)$, $u(t)$ such that

$$P([\ell(t), u(t)] \ni f(t) \ \ \forall t) \geq 1 - \alpha. \tag{10}$$

As noted above, basing the confidence set for f on a chi-square measure of misfit to the entire data vector δ is statistically inconsistent, in the sense that the confidence intervals do not shrink as the number of data increases. However, by averaging the data in adjacent bins and basing the confidence set on a norm of the misfit to those averages, one can construct consistent simultaneous confidence intervals. Furthermore, the optimization problems one must solve can be reduced without approximation to a sequence of finite-dimensional linear or quadratic programs (for the one, two, or infinity-norm), which can be solved efficiently and accurately in practice (Stark and Parker, Comp. Stat., 1995).

The dependence of the number of averages on the number of data determines a tradeoff between the variance of the averages, and the bias in the averages as estimates of the value of f at the endpoints of the intervals from the variation of f on the intervals. This is a particular instance of "teleological data compression" in which an optimal way to compress the data to sharpen inferences about α can be determined explicitly and fairly easily. I propose to apply this procedure to estimate the surface phase shift α from GONG and SOHO-SOI data as they become available. I will use the resulting confidence intervals for α as a function of ω in the asymptotic "strict bounds" inversion for soundspeed as a function of radius in the Sun, incorporating the uncertainty in the estimates of α into the uncertainty of the estimated soundspeed.

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